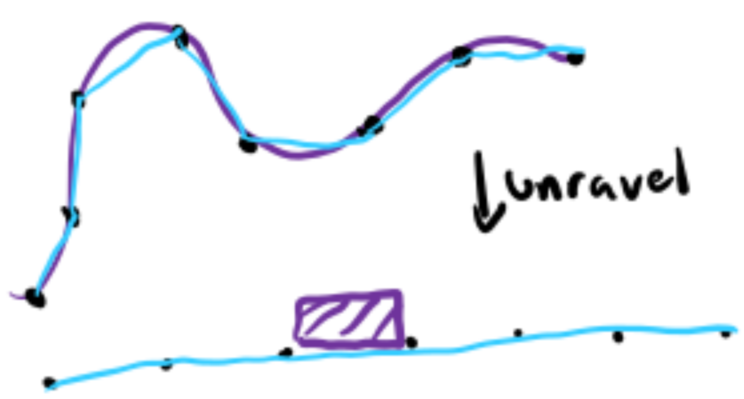
Lecture 11/8

Monday, November 8, 2021 9:39 AM

Section 16.2 and 16.3 - Line Integrals

Idea: Function with n variables and a curve in $\mathbb{R}^n$, understand how f builds up along the curve, $f: \mathbb{R}^n \to \mathbb{R}$, curve parametrized by interval $[a,b]$

$$\begin{array}{ccc} & \mathbb{R}^n & \\ \nearrow & & \searrow \\ [a,b] & \dashrightarrow & \mathbb{R} \\ & f(\vec{r}(t)) & \end{array}$$



Steps

1) Approximate curve using piecewise linear segments
2) unravel the approximation to an interval
3) use approximation to approximate the height $f(\ldots$ and width of segment length
4) limit these approximations by refining the segments

The line integral of a function f along a curve C parametrized by $\vec{r}(t)$ on $a \le t \le b$

$$\int_C f \underbrace{ds}_{\text{arc length}} = \int_a^b f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

Note: if $f=1$, then $S(c) = \int_C 1 \, ds = \int_a^b |\vec{r}\,'(t)| \, dt$ = arc length of C

Example: compute $\int_C f \, ds$ for $f(x,y) = x^2+y^2-xy$ and C, the upper hemisphere of the unit circle with $\underbrace{\text{positive orientation}}_{\text{Counterclockwise}}$

$$\int_C f \, ds = \int_{t=a}^{b} f(\vec{r}(t)) \, |\vec{r}(t)| \, dt$$

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}\,'(t)| = \sqrt{-\sin(t)^2 + \cos(t)^2} = 1$

$$\int_{t=0}^{\pi} (1 - \cos(t)\sin(t)) \cdot 1 \, dt$$

$$t - \frac{1}{2}\cos^2(t) \Big]^{\pi}$$

$u = \cos(t)$

$f\langle\cos(t), \sin(t)\rangle$

$f(\vec{r}(t)) = (\cos^2(t) + \sin^2(t)) - \cos(t)\sin(t)$

$$t - \frac{1}{2}\cos^2(t) \Big|$$

$u = \cos(t)$

$du = -\sin(t)\,dt$

$$\left(\pi - \frac{1}{2}(-1)^2\right) - \left(0 + \frac{1}{2}(1^2)\right)$$

$$= \boxed{\pi}$$

$$f(\vec{r}(t)) = (\cos^2(t) + \sin^2(t)) - \cos(t)\sin(t)$$

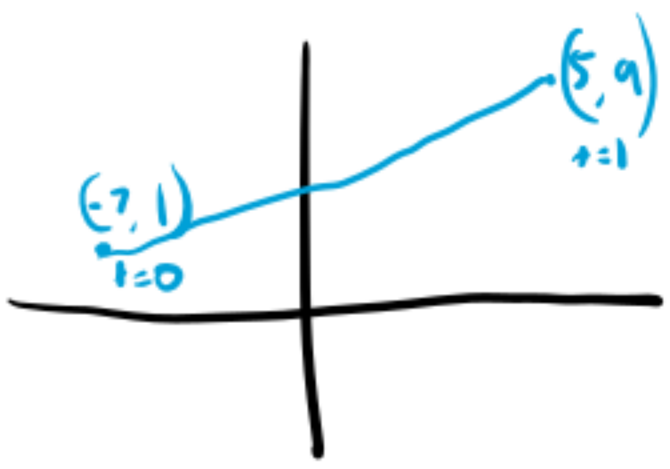
$$f(\vec{r}(t)) = 1 - \cos(t)\sin(t)$$

Directional line integral

For curve C parametrized by $f(\vec{r}(t))$ on $[a,b]$, and x_k as a variable of f,

$$\int_C f\,dx_k = \int_{t=a}^{b} f(\vec{r}(t)) \cdot \underbrace{x_k'}_{\substack{\text{derivative of the} \\ x_k \text{ component of } \vec{r}(t)}}(t)\,dt$$

Example: compute $\int_C y^2\,dx + \int_C x\,dy$ for C the line segment is oriented from $(-7,1)$ to $(5,9)$



$$\vec{r}(t) = (1-t)\langle -7, 1\rangle + t\langle 5, 9\rangle$$

$$\langle -7+12t,\ 1+8t\rangle$$

$$\vec{r}'(t) = \langle 12, 8\rangle$$

$$\int_C y^2\,dx + \int_C x\,dy$$

$$\int_{t=0}^{1} (1+8t)^2 \cdot 12\,dt + \int_{t=0}^{1} (-7+12t) \cdot 8\,dt$$

$$\int_{t=0}^{1} \left(12(1+16t+64t^2) + 8(12t-7)\right) dt$$

$$4\int_{t=0}^{1} 3 + 48t + 192t^2 + 24t - 14\,dt$$

$$4\int_{t=0}^{1} -11 + 72t + 192t^2\,dt$$

$$= 4\left[-11t + 36t^2 + 64t^3\right]_{t=0}^{1}$$

$$4(-11+36+64-0) = \boxed{356}$$

$$t=0$$

$$4(-11+36+64-0)=\boxed{356}$$

line Integral type 3

The line integral of vector field $\vec{v}$ along curve C parametrized by $\vec{r}(t)$ on $[A,B]$ is

$$\int_C \vec{v}\,dr = \int_{t=a}^{b} \vec{v}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

or

$$\int_C \vec{V}\cdot\vec{T}\,ds$$ where $\vec{T}(t)$ is the unit tangent of $\vec{r}(t)$ $\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Example: compute $\int_C \vec{v}\,dr$ for $\vec{v}=\langle xy, yz, xz\rangle$ and C is the curve parametrized by $\vec{r}(t)=\langle t, t^2, t^3\rangle$, $0\le t\le 2$

$$r'(t)=\langle 1, 2t, 3t^2\rangle$$

$$\vec{v}(\vec{r}(t))=\langle t^3, t^5, t^4\rangle$$

$$\int_C \vec{v}\,dr = \int_{t=a}^{b} \vec{v}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\int_{t=0}^{2} \langle t^3, t^5, t^4\rangle\cdot\langle 1, 2t, 3t^2\rangle$$

$$\int_0^2 t^3+2t^6+3t^6 = \frac{1}{4}t^4+\frac{5}{7}t^7\Big|_0^2$$

$$\frac{16}{4}+\frac{5}{7}\cdot 128 = \frac{668}{7}$$

From Physics, the work done within a particle along a curve C thorough a vector field is given by

$$\vec{F}=\int_C \vec{F}\cdot d\vec{r}$$

Exercise: Compute the work done by a particle moving along the unit circle counter clockwise for the quarter-circle through the full force by $\vec{F}=\langle y^2, -x\rangle$

Exercise: Compute the [illegible] along the quarter-circle through the full force by $\vec{F} = \langle y^2, -xy \rangle$

Note: $$\int_C P\,dx + Q\,dy = \int_C P\,dx + \int_C Q\,dy$$

is there an analog of the fundamental theorem of calculus for use with line integrals?

Bad news: the answer is no

Good news: when $\vec{v}$ is a conservative vector field, the potential function acts as an antiderivative

Fundamental Theorem of Line Integrals

Let $f: \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives and suppose C is a smooth curve in $\mathbb{R}^n$ parametrized by $\vec{r}(t)$ on $[a,b]$ then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int_C \nabla f \cdot d\vec{r}$$

$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\int_{t=a}^{b} \frac{d}{dt}\left[f(\vec{r}(t))\right] dt$$

FTC: $$\left[f(\vec{r}(t))\right]_{t=a}^{b} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Example: compute $\int_C \vec{v} \cdot d\vec{r}$ for $\vec{v} = \langle 1+xy, e^{xy}, x^2 e^{xy} \rangle$ on $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$ for $0 \le t \le \pi/2$

$$\frac{d}{dy}\left[(1+xy)e^{xy}\right] = (1+xy)\cdot xe^{xy} + (0+x)e^{xy} = e^{xy}(2x+x^2y+0)$$

$$\frac{d}{dx}\left[x^2e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = e^{xy}(2x+x^2y)$$

so, $\frac{d}{dy} = \frac{d}{dx}$

$$f(x,y) = \int \frac{\partial f}{\partial y}\,dy = \int x^2e^{xy}\,dy$$

$$= xe^{xy} + C(x)$$

$$(1+xy)e^{xy} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xe^{xy} + C(x)\right]$$

$$e^{xy} + xy\,e^{xy} + C'(x)$$

$$(1+xy)e^{xy} + C'(x)$$

$C'(x) = 0$ so $C(x) = \underbrace{D}_{\text{constant}}$

so, $f(x,y) = xe^{xy} + D$

is $D=0$

$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\left(\tfrac{\pi}{2}\right)\right) - f\left(\vec{r}(0)\right)$$

evaluate